Math 217 Fall 2025

Quiz 16 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Suppose A is an $m \times n$ matrix, the *transpose* of A is ...

   Solution: The $n \times m$ matrix A^{T} whose (j, i) entry equals the (i, j) entry of A:

   $$(A^{\mathsf{T}})_{ji} = a_{ij} \qquad (1 \leq i \leq m,\ 1 \leq j \leq n).$$

   Equivalently,

   $$(A^{\mathsf{T}})_{ij} = a_{ji} \qquad (1 \leq i \leq n,\ 1 \leq j \leq m).$$

   (b) Suppose V and W are vector spaces and $T : V \to W$ is a linear transformation. The *image* of T is ...

   Solution: The set of all vectors in W that are images of vectors in V under T:

   $$\operatorname{im} T \;=\; \{\, T(v) \;:\; v \in V \,\}.$$

   This set is a subspace of W, called the *image* of T.

   (c) Suppose U is a vector space and $u_1, \dots, u_n \in U$. The *span* of $(u_1, \dots, u_n)$ is ...

   Solution: The set of all finite linear combinations of the u_i:

   $$\operatorname{span}(u_1, \dots, u_n) \;=\; \Big\{ \sum_{i=1}^{n} a_i u_i \;:\; a_1, \dots, a_n \in \mathbb{F} \Big\},$$

   where $\mathbb{F}$ is the underlying field (e.g., $\mathbb{R}$ or $\mathbb{C}$).

2. Fix any ordered basis $(v_1, \dots, v_n)$ for V, and consider the map

   $$\phi : \mathbb{R}^n \to V, \qquad \phi\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1 v_1 + \cdots + a_n v_n.$$

   (a) Show that ϕ is a linear transformation.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

Solution: Let $x = (a_1, \ldots, a_n)^\mathsf{T}$ and $y = (b_1, \ldots, b_n)^\mathsf{T}$ in $\mathbb{R}^n$, and $c \in \mathbb{F}$.

$$\phi(x+y) = \sum_{i=1}^{n} (a_i + b_i) v_i = \sum_{i=1}^{n} a_i v_i + \sum_{i=1}^{n} b_i v_i = \phi(x) + \phi(y),$$

$$\phi(cx) = \sum_{i=1}^{n} (c\, a_i) v_i = c \sum_{i=1}^{n} a_i v_i = c\, \phi(x).$$

Thus ϕ is linear.

(b) Show that ϕ is an *isomorphism*.

Solution: Since $(v_1, \ldots, v_n)$ is a basis, every $v \in V$ has a unique coordinate vector $(a_1, \ldots, a_n)^\mathsf{T}$ with $v = \sum_{i=1}^{n} a_i v_i$. This shows:

- *Surjectivity:* For any $v \in V$, choose its coordinates a_i and get $\phi((a_1, \ldots, a_n)^\mathsf{T}) = v$.
- *Injectivity:* If $\phi(x) = \phi(y)$, then $\sum_{i=1}^{n} (a_i - b_i) v_i = 0$. By linear independence of the basis, $a_i - b_i = 0$ for all i, so $x = y$.

Therefore ϕ is bijective and linear, hence an isomorphism.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) If V is a vector space and $\mathcal{S}$ is a finite list of vectors in V such that $\vec{0}$ is on the list, then $\mathcal{S}$ is linearly dependent.

Solution: TRUE. If 0_V is in the list, take coefficients 1 for that entry and 0 for all others; then a nontrivial linear combination equals 0_V. Hence the list is linearly dependent.

(b) Any four vectors in $\mathbb{R}^3$ are linearly dependent.

Solution: TRUE. The dimension of $\mathbb{R}^3$ is 3, so any list with more than 3 vectors is linearly dependent (no more than dim many can be linearly independent).